Math 451: Introduction to General Topology
Lecture 9

Before we prove the existence of a completion, we need to discuss pseudo-metric spaces.

Def. A pseudo-metric on a set X is a function $d: X \times X \to [0,\infty)$ satisfying:

(i) $d(x,x) = 0 \quad \forall x \in X$.

(ii) $d(x,y) = d(x,y) \quad \forall x,y \in X$.

(iii) Triangle inequality: $d(x,z) \leq d(x,y) + d(y,z) \quad \forall x,y,z \in X$.

Call (X,d) a pseudo-metric space.

Given a pseudo-metric space, define an equivalence relation $\sim$ on X by
$$x \sim y \;:\Leftrightarrow\; d(x,y) = 0.$$
Then we can turn the quotient $\hat{X} := X/\sim$ into a metric space as follows. Let $[x]$ denote the equivalence class of $x \in X$, so $[x] = \{y \in X : d(x,y) = 0\}$. Then define $\tilde{d}: \tilde{X} \times \tilde{X} \to [0,\infty)$ by $\tilde{d}([x],[y]) := d(x,y)$. This is well-defined, i.e. if $x' \sim x$ and $y' \sim y$ then $d(x,y) = d(x',y')$, which holds by Δ-inequality:
$$d(x,y) \leq d(x,x') + d(x',y') + d(y',y) = d(x',y')$$
and $\geq$ is by symmetry. The triangle inequality is easy to verify, so $\tilde{d}$ is a metric on $\tilde{X}$. Then the quotient map $\pi: X \to X/\sim =: \tilde{X}$ is an "isometry": for all $x,y \in X$,
$$\tilde{d}(\pi(x), \pi(y)) = d(x,y).$$

Proof of the existence of completion. Let (X,d) be a metric space. Let $\mathrm{Cauchy}(X) :=$ all Cauchy sequences in X, in particular, $\mathrm{Cauchy}(X) \subseteq X^{\mathbb{N}}$. We will make this a pseudo-metric space whose quotient metric space will be our completion $(\hat{X},\hat{d})$.

Define $\mathcal{D}: \mathrm{Cauchy}(X)^2 \to [0,\infty)$ by
$$\mathcal{D}((x_n),(y_n)) := \lim_{n\to\infty} d(x_n,y_n).$$

Claim 1. The sequence $(d(x_n,y_n)) \subseteq \mathbb{R}$ is Cauchy and therefore has a limit in $\mathbb{R}$.

Proof. For each $\varepsilon>0$ fix $N\in\mathbb{N}$ s.t. $\forall n,m \geq N$, $d(x_n,x_m)<\varepsilon/2$ and $d(y_n,y_m)<\varepsilon/2$. Then

$d(x_n,y_n) \leq d(x_n,x_m)+d(x_m,y_m)+d(y_m,y_n) \leq \frac{\varepsilon}{2}+d(x_m,y_m)+\frac{\varepsilon}{2} = d(x_m,y_m)+\varepsilon$, and

$d(x_m,y_m) \leq d(x_m,x_n)+d(x_n,y_n)+d(y_n,y_m) \leq \varepsilon/2 + d(x_n,y_n)+\varepsilon/2 = d(x_n,y_n)+\varepsilon$, so

$$-\varepsilon \leq d(x_n,y_n)-d(x_m,y_m) \leq \varepsilon,$$

i.e. $|d(x_n,y_n)-d(x_m,y_m)| \leq \varepsilon$, so indeed $(d(x_n,y_n)) \subseteq \mathbb{R}$ is Cauchy. $\square$

It is quick to verify that the triangle inequality holds for $\mathcal{D}$, making it a pseudo-metric on $\mathrm{Cauchy}(X)$. Let $(\hat{X},\hat{d})$ be the quotient metric space. We note that (X,d) isometrically injects into $\hat{X}$ by $\begin{aligned} X &\hookrightarrow \hat{X} \\ x &\mapsto [(x,x,x,\dots)] \end{aligned}$; indeed, for any $x,y\in X$,

$\hat{d}([(x,x,\dots)],[(y,y,\dots)]) := \mathcal{D}\big((x,x,\dots),(y,y,\dots)\big) = \lim\limits_{n\to\infty} d(x,y) = d(x,y)$. It remains to show that $(\hat{X},\hat{d})$ is complete and we do not need to verify that $\overline{X}^{\hat{d}} = \hat{X}$ because if not, then the completion of X would be $\overline{X}^{\hat{d}}$. (Although it is true that $\overline{X}^{\hat{d}} = \hat{X}$. HW)

Claim 2. For any Cauchy $(x_n) \subseteq X$ and any subsequence (x_{n_k}), $(x_n) \sim (x_{n_k})$, i.e.

$$\mathcal{D}((x_k),(x_{n_k})) = \lim_{k\to\infty} d(x_k,x_{n_k}) = 0$$

Proof. For any $\varepsilon>0$ let N be such that $\operatorname{diam}\{x_N,x_{N+1},\dots\}<\varepsilon$, then $\forall k\geq N$, we have $n_k \geq N$ and $d(x_k,x_{n_k}) \leq \operatorname{diam}\{x_N,x_{N+1},\dots\}<\varepsilon$. $\square$

This says that when given $[(x_n)]\in\hat{X}$, we can move to a faster subsequence (x_{n_k}) within the same equivalence class $[(x_n)]$.

To prove that $(\hat{X},\hat{d})$ is complete, take a Cauchy sequence $(\hat{x}^k)_{k\in\mathbb{N}} \subseteq \hat{X}$. By moving to a fast subsequence of $(\hat{x}^k)$ and then choosing a fast representative $(x^k_n)_{n\in\mathbb{N}}$ for each $k\in\mathbb{N}$ and $n\geq k$, we may assume

(i) $d(x^k_k, x^k_n) < \frac{1}{k}$;

(ii) $d(x^k_n, x^n_n) < \frac{1}{k}$.

$$\begin{array}{llllll}
\hat{x}^1: & \boxed{x^1_1} & x^1_2 & x^1_3 & x^1_4 & x^1_5 \;\dots \\
\hat{x}^2: & x^2_1 & \boxed{x^2_2} & x^2_3 & x^2_4 & x^2_5 \;\dots \\
\hat{x}^3: & x^3_1 & x^3_2 & \boxed{x^3_3} & x^3_4 & x^3_5 \;\dots \\
\hat{x}^4: & x^4_1 & x^4_2 & x^4_3 & \boxed{x^4_4} & x^4_5 \;\dots \\
 & & \vdots & & &
\end{array}$$

It follows using (i) and (ii) that the diagonal sequence (x_n^n) is Cauchy, so $\hat{x} := [(x_n^n)] \in \hat{X}$. One then checks, again using (i) and (ii) that for any $n \geq k$, $d(x_n^k, x_n^n) \leq \frac{1}{k}$, so $\lim_{n\to\infty} d(x_n^k, x_n^n) \leq \frac{1}{k}$, so $\mathcal{D}((x_n^k), (x_n^n)) \leq \frac{1}{k}$, hence $\hat{d}(\hat{x}_k, x) \leq \frac{1}{k}$. We leave these verification as exercise. $\square$

We will give a much cleaner, although less intuitive, alternative proof of the existence of a completion by Kaplanski shortly.

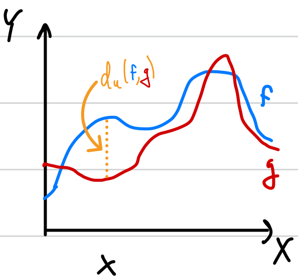
Remark. For a metric space X and a subset $Y \subseteq X$, we say that Y is dense in X if Y meets every nonempty open set U (i.e. $\exists y \in Y$ s.t. $y \in U$, like $k \cap B$). This is equivalent (HW) to $\overline{Y} = X$. Thus, a completion $\hat{X}$ of X is a metric space with $X \subseteq \hat{X}$ where X is dense in $\hat{X}$ and $\hat{X}$ is complete.

To give Kaplanski's proof, we need the following metric space.

Space of functions with uniform metric.

For a set X and a metric space Y, let $B(X,Y)$ denote the set of bounded functions $X \to Y$, where a function $f: X \to Y$ is called bounded if $f(X) \subseteq$ some ball in Y. In this case, we can define the uniform metric between $f, g \in B(X,Y)$

$$d_u(f,g) := \sup_{x \in X} d_Y(f(x), g(x)).$$



This is clearly a metric; indeed, $d_u(f,g) = 0$ implies $f(x) = g(x)$ $\forall x \in X$, hence $f = g$. Also,

$$d_u(f,h) = \sup_{x \in X} d_Y(f(x), h(x)) \leq \sup_{x \in X} \left(d_Y(f(x), g(x)) + d_Y(g(x), h(x)) \right)$$

$$\leq \sup_{x_0 \in X} d_Y(f(x_0), g(x_0)) + \sup_{x_1 \in X} d_Y(g(x_1), h(x_1)) = d_u(f,g) + d_u(g,h).$$

Theorem. If (Y, d_Y) is a complete metric space then $B(X, Y)$ is complete with the uniform metric d_u.

Proof. Let (f_n) be a d_u-Cauchy sequence. Note that for each $x \in X$, $(f_n(x))_{n\in\mathbb{N}} \subseteq Y$ is d_Y-Cauchy because $d_Y(f_n(x), f_m(x)) \leq d_u(f_n, f_m) \to 0$ as $n, m \to \infty$. By the completeness of Y, the limit of $(f_n(x))$ exists and we put $f(x) := \lim_{n\to\infty} f_n(x)$. We need to show that f is bdd and $f_n \to_{d_u} f$ as $n \to \infty$.

Claim 1. f is bdd.

Proof. By Cauchy property, $\exists N \in \mathbb{N}\ \forall n, k \geq N$ $d_u(f_n, f_k) \leq 1$. But $\forall n$, $f_n(X) \subseteq B_{r_n}(y_n)$ for some $y_n \in Y$, so setting $r := \max\{d(y_0, y_n) + r_n : n \leq N\} + 1$, we get that for all $n \in \mathbb{N}$, $f_n(X) \subseteq B_r(y_0)$. □

Claim 2. (f_n) converges to f in the uniform metric (i.e. converges uniformly).

Proof. Fix an arbitrary $\varepsilon > 0$. By the Cauchyness of (f_n) in the uniform distance, $\exists N \in \mathbb{N}$ s.t. $\forall n, k \geq N$, (*) $d_u(f_n, f_k) < \varepsilon/2$. Now we fix $n \geq N$, aiming to show that $d_u(f, f_n) \leq \varepsilon$. To do so, we show that for each $x \in X$, $d_Y(f(x), f_n(x)) \leq \varepsilon$, so fix $x \in X$. Because $\lim_{k\to\infty} f_k(x) = f(x)$, there is $k \geq N$ such that $d_Y(f(x), f_k(x)) \leq \varepsilon/2$ (⁑). Thus,

$$d_Y(f(x), f_n(x)) \leq d(f(x), f_k(x)) + d_Y(f_k(x), f_n(x)) \underset{(⁑)}{\leq} \varepsilon/2 + \underset{(*)}{\varepsilon/2} = \varepsilon.$$

Thus, $d_u(f, f_n) = \sup_{x\in X} d_Y(f(x), f_n(x)) \leq \varepsilon$, so $f_n \to_{d_u} f$ as $n \to \infty$. □

QED